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Examples of non-archimedean nuclear Fréchet spaces without a Schauder basis

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ABSTRACT

We solve the problem of the existence of a Schauder basis in non-archimedean Fréchet spaces of countable type (stated in [3]). Using examples of real nuclear Fréchet spaces without a Schauder basis (of Bessaga [1], Mitiagin [5] and Vogt [10]) we construct examples of non-archimedean nuclear Fréchet spaces without a Schauder basis (even without the bounded approximation property).

1. INTRODUCTION

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0, \infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [8], [6] and [7]. Schauder bases in locally convex spaces are studied in [2], [3], [4] and [9].

Any infinite-dimensional Banach space of countable type is linearly homeomorphic to the Banach space c_0 of all sequences in $\mathbb{K}$ converging to zero (with the sup-norm) ([7], Theorem 3.16), so it has a Schauder basis. It is also known that any metrizable lcs of finite type has a Schauder basis ([3], Theorem 3.5). In [9] we proved that any infinite-dimensional metrizable lcs contains an infinite-dimensional closed subspace with a Schauder basis.

In this paper we solve the problem stated in [3], whether any Fréchet space of countable type has a Schauder basis. We show that there exist nuclear Fréchet spaces without a Schauder basis. First, we construct an infinite family of pair-wise-nonisomorphic nuclear Fréchet spaces with a strongly finite-dimensional

Schauder decomposition but without a Schauder basis (see Theorem 3 and Corollary 5). Next, we give two examples of nuclear Fréchet spaces with a finite-dimensional Schauder decomposition but without a strongly finite-dimensional Schauder decomposition (see Theorem 7 and Corollary 9). Finally, we present an example of a nuclear Fréchet space with a Schauder decomposition but without a finite-dimensional Schauder decomposition (even without the bounded approximation property) (see Theorem 11). Our examples are non-archimedean modifications of the real nuclear Fréchet spaces without a Schauder basis constructed by Bessaga [1], Mitiagin [5], and Vogt [10].

2. PRELIMINARIES

The linear span of a subset A of a linear space E is denoted by $\mathrm{lin}A$.

The linear space of all continuous linear operators from a lcs E to itself will be denoted by $\mathcal{L}(E)$.

A sequence (x_n) in a lcs E is a *Schauder basis* of E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $\alpha_n \in \mathbb{K}, n \in \mathbb{N}$, and the coefficient functionals $f_n : E \to \mathbb{K}, x \to \alpha_n (n \in \mathbb{N})$ are continuous.

A sequence (P_n) of continuous linear non-zero projections on a lcs E is a *Schauder decomposition* of E if $x = \sum_{n=1}^{\infty} P_n x$ for all $x \in E$ and $P_n P_m = 0$ for all $n \neq m$.

A Schauder decomposition (P_n) of a lcs E is *finite-dimensional* if $\dim P_n(E) < \infty$ for $n \in \mathbb{N}$, and *strongly finite-dimensional* if $\sup_n \dim P_n(E) < \infty$. Clearly, any lcs E with a Schauder basis has a strongly finite-dimensional Schauder decomposition.

A lcs E has the *bounded approximation property* if there exists a sequence $(A_n) \subset \mathcal{L}(E)$ with $\dim A_n(E) < \infty$ for $n \in \mathbb{N}$ such that $\lim_n A_n x = x$ for all $x \in E$. Of course any lcs E with a finite-dimensional Schauder decomposition has the bounded approximation property.

By a *seminorm* on a linear space E we mean a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x + y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if $\mathrm{Ker}\, p := \{x \in E : p(x) = 0\} = \{0\}$.

Two norms p, q on a linear space E are *equivalent* if there exist positive numbers a, b such that $ap(x) \leq q(x) \leq bp(x)$ for every $x \in E$. Every two norms on a finite-dimensional linear space are equivalent.

Every n-dimensional lcs is linearly homeomorphic to the Banach space $\mathbb{K}^n$.

A lcs E is *of finite type* if for each continuous seminorm p on E the quotient space $E/\mathrm{Ker}\, p$ is finite-dimensional. A metrizable lcs E is *of countable type* if it contains a linearly dense countable set.

A *Fréchet space* is a metrizable complete lcs.

Any non-decreasing sequence $(\|\cdot\|_n)$ of norms on a linear space E defines a metrizable locally convex linear topology on E. This metrizable lcs will be denoted by $(E, (\|\cdot\|_n))$. The sets $\{x \in E : \|x\|_n \leq m^{-1}\}, n, m \in \mathbb{N}$ form a base of neighbourhoods of 0 in $(E, (\|\cdot\|_n))$. A seminorm p on E is continuous iff there exist $m \in \mathbb{N}$ and $c > 0$ such that $p(x) \leq c\|x\|_m$ for all $x \in E$.

A subset B of a lcs E is *compactoid* if for each neighbourhood U of 0 in E there exists a finite subset $A = \{a_1, \ldots, a_n\}$ of E such that $B \subset U + \mathrm{co}\, A$, where $\mathrm{co}\, A = \{\sum_{i=1}^{n} \alpha_i a_i : \alpha_1, \ldots, \alpha_n \in \mathbb{K}, |\alpha_1|, \ldots, |\alpha_n| \leq 1\}$ is the *absolutely convex hull* of A.

Let E and F be locally convex spaces. The linear map $T : E \to F$ is *compact* if there exists a neighbourhood U of 0 in E such that $T(U)$ is compactoid in F.

For any seminorm p on a lcs E the map $\bar{p} : E/\mathrm{Ker}\,p \to [0, \infty), x + \mathrm{Ker}\,p \to p(x)$ is a norm on $E/\mathrm{Ker}\,p$.

A lcs E is *nuclear* if for every continuous seminorm p on E there exists a continuous seminorm q on E with $q \geq p$ such that the map

$$\varphi_{pq} : (E/\mathrm{Ker}\,q, \bar{q}) \to (E/\mathrm{Ker}\,p, \bar{p}),\ x + \mathrm{Ker}\,q \to x + \mathrm{Ker}\,p$$

is compact. Any nuclear lcs is of countable type ([8], Corollary 4.14).

3. RESULTS

First we construct an infinite family of pairwise-nonisomorphic nuclear Fréchet spaces with a strongly finite-dimensional Schauder decomposition but without a Schauder basis (cf [1], [5]).

Put $\mathbb{N}_0 = \{r \in \mathbb{N} : r > 1 \text{ and } r\mathbf{1} \neq 0 \text{ in } \mathbb{K}\}$, where $\mathbf{1}$ is the unit element of $\mathbb{K}$. Clearly, the set $\mathbb{N}_0$ is infinite. Let $r \in \mathbb{N}_0$. Let $\{e_1, \ldots, e_r\}$ be a basis of the linear space $\mathbb{K}^r$ and let $e_1^*, \ldots, e_r^*$ be the coefficient functionals of this basis. Put $f_1 = \sum_{i=1}^{r} e_i, f_1^* = \sum_{i=1}^{r} e_i^*$ and $f_j^* = e_{j-1}^* - e_j^*$ for $2 \leq j \leq r$. It is easy to see that

$$e_1^* = (r\mathbf{1})^{-1}[f_1^* + \sum_{i=2}^{r}(\sum_{j=2}^{i} f_j^*)] \text{ and } e_i^* = (e_1^* - \sum_{j=2}^{i} f_j^*)\ , 2 \leq i \leq r.$$

Hence $|e_j^*(x)| \leq |r\mathbf{1}|^{-1} \max_l |f_l^*(x)|$ for all $x \in \mathbb{K}^r, 1 \leq j \leq r$.

Let $n \in \mathbb{N}$. Consider a finite sequence $(|\cdot|_{n,j}^r)_{j=1}^{r+1}$ of norms on the space $\mathbb{K}^r$:

$$|x|_{n,j}^r = \begin{cases} \max(\{2^{n(j-1)}|e_j^*(x)|\} \cup \{2^{nj}|e_i^*(x)| : i \neq j\}) & \text{if } 1 \leq j \leq r, \\ |r\mathbf{1}|^{-1} \max(\{2^{nr}|f_1^*(x)|\} \cup \{2^{n(r+1)}|f_l^*(x)| : l > 1\}) & \text{if } j = r+1. \end{cases}$$

Clearly, $|x|_{n,j}^r \leq |x|_{n,j+1}^r$ for all $x \in \mathbb{K}^r, 1 \leq j \leq r$.

Let $\beta \in \mathbb{K}$ with $|\beta| > 1$. Set $d = |\beta|$. For a linear operator $T : \mathbb{K}^r \to \mathbb{K}^r$, $z^* \in (\mathbb{K}^r)^*$ and $j \leq r+1$ we put $\|T\|_{n,j}^r = \sup\{|Tx|_{n,j}^r : x \in \mathbb{K}^r, |x|_{n,j}^r \leq 1\}$ and $\|z^*\|_{n,j}^r = \sup\{|z^*(x)| : x \in \mathbb{K}^r, |x|_{n,j}^r \leq 1\}$. Then $|Tx|_{n,j}^r \leq d\|T\|_{n,j}^r |x|_{n,j}^r$ and $|z^*(x)| \leq d\|z^*\|_{n,j}^r |x|_{n,j}^r$ for all $x \in \mathbb{K}^r, j \leq r+1$.

We will need the following

Lemma 1. *Let* $(T_k) \subset \mathcal{L}(\mathbb{K}^r)$. *If* $\max_k \dim T_k(\mathbb{K}^r) < r$ *and* $\sum_{k=1}^{\infty} T_k x = x, x \in \mathbb{K}^r$, *then* $\max\{\|T_k\|_{n,j}^r : k \in \mathbb{N}, j \leq r+1\} \geq 2^n d^{-2}$.

Proof. Let $k \in \mathbb{N}$. First we show that

$$(*) \qquad |e_1^* T_k e_1| \leq \max\{\max_{i \neq j} |e_i^* T_k e_j|, \max_{l>1} |f_l^* T_k f_1|\}.$$

Let $t_{i,j} = e_j^* T_k e_i$ for $1 \leq i,j \leq r$. Since $\dim T_k(\mathbb{K}^r) < r$, then there exist $\alpha_1, \ldots, \alpha_r \in \mathbb{K}$ and $1 \leq j_0 \leq r$ such that $\sum_{j=1}^r \alpha_j T_k e_j = 0$ and $\max_j |\alpha_j| = |\alpha_{j_0}| > 0$. Hence $|t_{j_0,j_0}| = |-\sum_{j \neq j_0} \alpha_{j_0}^{-1} \alpha_j t_{j,j_0}| \leq \max_{j \neq j_0} |t_{j,j_0}|$. This yields $(*)$, if $j_0 = 1$.

If $j_0 > 1$, then we have

$$|t_{1,1}| = |t_{j_0,j_0} + \sum_{i \neq j_0} t_{i,j_0} - \sum_{i \neq 1} t_{i,1} + (\sum_i t_{i,1} - \sum_i t_{i,j_0})| =$$

$$|t_{j_0,j_0} + \sum_{i \neq j_0} t_{i,j_0} - \sum_{i \neq 1} t_{i,1} + \sum_{l=2}^{j_0} f_l^* T_k f_1| \leq$$

$$\max\{\max_{i \neq j} |e_i^* T_k e_j|, \max_{l>1} |f_l^* T_k f_1|\}.$$

This proves $(*)$.

For $1 \leq i,j \leq r, i \neq j$ we have $|e_i^* T_k e_j| \leq d^2 \|e_i^*\|_{n,j}^r \|T_k\|_{n,j}^r |e_j|_{n,j}^r \leq$

$$d^2 2^{-nj} \|T_k\|_{n,j}^r 2^{n(j-1)} = 2^{-n} d^2 \|T_k\|_{n,j}^r.$$

For $2 \leq l \leq r$ we obtain $|f_l^* T_k f_1| \leq d^2 \|f_l^*\|_{n,r+1}^r \|T_k\|_{n,r+1}^r |f_1|_{n,r+1}^r \leq$

$$d^2 2^{-n(r+1)} |r1| \, \|T_k\|_{n,r+1}^r 2^{nr} \leq 2^{-n} d^2 \|T_k\|_{n,r+1}^r.$$

Thus $|e_1^* T_k e_1| \leq 2^{-n} d^2 \max\{\|T_k\|_{n,j}^r : 1 \leq j \leq r+1\}, k \in \mathbb{N}$. Hence we obtain

$$\begin{aligned} 1 &= \left| \sum_{k=1}^{\infty} e_1^* T_k e_1 \right| \\ &\leq \max_k |e_1^* T_k e_1| \leq 2^{-n} d^2 \max\{\|T_k\|_{n,j}^r : k \in \mathbb{N}, 1 \leq j \leq r+1\}. \end{aligned}$$

This completes the proof of Lemma 1. $\square$

Put $\Psi_r = \{(p_1, \ldots, p_r) \in \mathbb{N}^r : p_1 < p_2 < \cdots < p_r\}$. Let $\sigma_r : \mathbb{N} \to \Psi_r$ be an infinity-to-one surjection (it means that the inverse image of every singleton is infinite).

Let $n \in \mathbb{N}$. Set $p_0 = 0$ and $\sigma_r(n) = (p_1, \ldots, p_r)$. Consider the following sequence $(\|\cdot\|_{n,k}^r)_{k=1}^\infty$ of norms on the space $X_n^r = \mathbb{K}^r$:

$$\|x\|_{n,k}^r = \begin{cases} |x|_{n,j+1}^r & \text{if } p_j < k \leq p_{j+1}, \ 0 \leq j \leq r-1, \\ |x|_{n,r+1}^r & \text{if } k > p_r. \end{cases}$$

Clearly, $\|x\|_{n,k}^r \leq \|x\|_{n,k+1}^r$ for $x \in X_n^r, k \in \mathbb{N}$.

Let $|||x|||_k^r = \sup_n n^k \|x_n\|_{n,k}^r$ for $x = (x_n) \in \prod_{n=1}^\infty X_n^r, k \in \mathbb{N}$ and

$$X^r = \left\{ x \in \prod_{n=1}^\infty X_n^r : |||x|||_k^r < \infty \text{ for all } k \in \mathbb{N} \right\}.$$

Clearly, $(|||\cdot|||_k^r)_{k=1}^\infty$ is a non-decreasing sequence of norms on the linear space X^r.

We have the following.

Proposition 2. *The metrizable lcs $X^r = (X^r, (|||\cdot|||_k^r)_{k=1}^\infty)$ is a nuclear Fréchet space with a strongly finite-dimensional Schauder decomposition.*

Proof. First we prove that X^r is a Fréchet space. Let (x^m) be a Cauchy sequence in X^r and $x^m = (x_n^m)_{n=1}^\infty, m \in \mathbb{N}$. Then

$$(*) \qquad \forall k \in \mathbb{N}\, \forall \epsilon > 0\, \exists M(\epsilon, k) > 0\, \forall m, l > M(\epsilon, k)\, \forall n \in \mathbb{N} : n^k \|x_n^m - x_n^l\|_{n,k}^r \le \epsilon.$$

Hence, for every $n \in \mathbb{N}, (x_n^m)_{m=1}^\infty$ is a Cauchy sequence in $(X_n^r, \|\cdot\|_{n,1}^r)$. Thus $\lim_m \|x_n^m - x_n^0\|_{n,1}^r = 0$ for some $x_n^0 \in X_n^r$. Then $\lim_m \|x_n^m - x_n^0\|_{n,k}^r = 0$ for all $k \in \mathbb{N}$, since $\dim X_n^r < \infty$. By $(*)$ we obtain

$$(**) \qquad \forall k \in \mathbb{N}\, \forall \epsilon > 0\, \forall m > M(\epsilon, k)\, \forall n \in \mathbb{N} : n^k \|x_n^m - x_n^0\|_{n,k}^r \le \epsilon.$$

Hence $\forall k \in \mathbb{N}\, \forall m > M(1,k) : |||x^0|||_k^r \le \max\{|||x^m|||_k^r, |||x^m - x^0|||_k^r\} < \infty$, where $x^0 = (x_n^0)$. Thus $x^0 \in X^r$. By $(**)$, we have $\lim_m |||x^m - x^0|||_k^r = 0$ for every $k \in \mathbb{N}$, so $x^m \to x^0$ in X^r.

To prove that X^r is nuclear, it is enough to show that for all $k \in \mathbb{N}, \epsilon > 0$ there exists a finite subset $A_k^r(\epsilon)$ of X^r such that $B_{k+1}^r(1) \subset B_k^r(\epsilon) + \operatorname{co} A_k^r(\epsilon)$, where $B_{k+1}^r(1) = \{x \in X^r : |||x|||_{k+1}^r \le 1\}$ and $B_k^r(\epsilon) = \{x \in X^r : |||x|||_k^r \le \epsilon\}$.

Let $k \in \mathbb{N}, \epsilon > 0$ and $m \in \mathbb{N}$ with $m > \epsilon^{-1}$. For $n \in \mathbb{N}$ let $\{e_{n,1}^r, \ldots, e_{n,r}^r\}$ be a basis in X_n^r with $\|\sum_{i=1}^r \alpha_i e_{n,i}^r\|_{n,k+1}^r \ge \max_{1 \le i \le r} |\alpha_i|$ for all $\alpha_1, \ldots, \alpha_r \in \mathbb{K}$.

Let $x = (x_n) \in B_{k+1}^r(1)$ and $x_n = \sum_{i=1}^r \alpha_{n,i}^r e_{n,i}^r, n \in \mathbb{N}$. Since $\|x_n\|_{n,k+1}^r \le 1$, then $x_n \in \operatorname{co}\{e_{n,1}^r, \ldots, e_{n,r}^r\}, n \in \mathbb{N}$. Put $f_{n,i}^r = (0, \ldots, 0, e_{n,i}^r, 0, \ldots) \in X^r$, where $e_{n,i}^r$ is on n-th place and $1 \le i \le r, n \in \mathbb{N}$.

Then $A_k^r(\epsilon) := \{f_{n,i}^r : n < m, i \le r\} \subset X^r$ and $(x_1, \ldots, x_{m-1}, 0, 0, \ldots) \in \operatorname{co} A_k^r(\epsilon)$. Since $n^k \|x_n\|_{n,k}^r \le n^{-1} |||x|||_{k+1}^r$ for $n \in \mathbb{N}$, then $(0, \ldots, 0, x_m, x_{m+1}, \ldots) \in B_k^r(\epsilon)$. Thus $x \in B_k^r(\epsilon) + \operatorname{co} A_k^r(\epsilon)$. Hence $B_{k+1}^r(1) \subset B_k^r(\epsilon) + \operatorname{co} A_k^r(\epsilon)$.

For $m \in \mathbb{N}$ we put $P_m : X^r \to X^r, (x_n) \to (0, \ldots, 0, x_m, 0, \ldots)$, where x_m is on m-th place. Clearly, $P_m, m \in \mathbb{N}$, are continuous linear projections with $\dim P_m(X^r) = r$ and $P_m P_l = 0$ for $m \ne l$. Let $x \in X^r$ and $k \in \mathbb{N}$. Since $n^k \|x_n\|_{n,k}^r \le n^{-1} |||x|||_{k+1}^r, n \in \mathbb{N}$, then $\lim_n n^k \|x_n\|_{n,k}^r = 0$. Hence $\sum_{m=1}^\infty P_m x = x, x \in X^r$. Thus X^r has a strongly finite-dimensional Schauder decomposition. $\square$

Now we prove that X^r has no Schauder basis.

Theorem 3. *For any sequence $(Q_k) \subset \mathcal{L}(X^r)$ such that $\sum_{k=1}^\infty Q_k x = x, x \in X^r$, there exists $k \in \mathbb{N}$ with $\dim Q_k(X^r) \ge r$. In particular, the space X^r has no Schauder basis.*

Proof. Suppose, by contradiction, that $\dim Q_k(X^r) < r$ for any $k \in \mathbb{N}$. Since the sequence (Q_k) is pointwise bounded, then, by the Banach-Steinhaus theorem ([6], Theorem 3.37), the operators $Q_k, k \in \mathbb{N}$, are equicontinuous. Hence for every continuous seminorm p on X^r the seminorm $q: X^r \to [0,\infty), x \to \max_k p(Q_k x)$ is continuous on X^r. Thus there exist integers $p_0 = 0 < p_1 < \cdots < p_r < p_{r+1}$ and a constant C such that $\max_k |||Q_k x|||^r_{p_i+1} \leq C|||x|||^r_{p_{i+1}}$ for $0 \leq i \leq r$.

Let $n \in \sigma_r^{-1}(\{(p_1, \ldots, p_r)\})$. Let $J_n : X_n^r \to X^r$ be the natural embedding and $P_n : X^r \to X_n^r$ the natural projection. Then $|||J_n x|||^r_k = n^k \|x\|^r_{n,k}$ for $x \in X_n^r$, $k \in \mathbb{N}$, and $\|P_n x\|^r_{n,k} \leq n^{-k}|||x|||^r_k$ for $x \in X^r, k \in \mathbb{N}$. Put $T_k = P_n Q_k J_n$ for $k \in \mathbb{N}$. Then $(T_k) \subset \mathcal{L}(X_n^r)$, $\max_k \dim T_k(X_n^r) < r$ and $\sum_{k=1}^{\infty} T_k x = x$ for all $x \in X_n^r$. For $0 \leq i \leq r, x \in X_n^r$ we have

$$\max_k \|T_k x\|^r_{n,p_i+1} = \max_k \|P_n Q_k J_n x\|^r_{n,p_i+1} \leq \max_k n^{-p_i-1}|||Q_k J_n x|||^r_{p_i+1} \leq$$
$$n^{-p_i-1}C|||J_n x|||^r_{p_{i+1}} = Cn^{p_{i+1}-p_i-1}\|x\|^r_{n,p_{i+1}} \leq Cn^{p_{i+1}}\|x\|^r_{n,p_{i+1}}.$$

Since $\|x\|^r_{n,p_i+1} = \|x\|^r_{n,p_{i-1}} = |x|^r_{n,i+1}$ for $x \in X_n^r$, $0 \leq i \leq r$, then

$$\max_k |T_k x|^r_{n,j} \leq Cn^{p_{r+1}}|x|^r_{n,j} \text{ for all } x \in X_n^r, 1 \leq j \leq r+1.$$

Hence $\max\{\|T_k\|^r_{n,j} : k \in \mathbb{N}, 1 \leq j \leq r+1\} \leq Cn^{p_{r+1}}$. Using Lemma 1 we obtain $Cn^{p_{r+1}} \geq 2^n d^{-2}$. Thus $2^n n^{-p_{r+1}} \leq Cd^2$ for every n in the infinite set $\sigma_r^{-1}(\{(p_1, \ldots, p_r)\})$. Since $\lim_n 2^n n^{-p_{r+1}} = \infty$, we get a contradiction. □

Corollary 4. *Let Y be a Fréchet space. For any sequence $(Q_k) \subset \mathcal{L}(X^r \times Y)$ such that $\sum_{k=1}^{\infty} Q_k z = z, z \in X^r \times Y$, there exists $k \in \mathbb{N}$ with* $\dim Q_k(X^r \times Y) \geq r$.

In particular, the Fréchet space $X^r \times Y$ has no Schauder basis.

Proof. Let $P : X^r \times Y \to X^r, (x,y) \to x$, and $S : X^r \to X^r \times Y, x \to (x,0)$. Put $Q_k^0 = PQ_kS$ for $k \in \mathbb{N}$. Then $(Q_k^0) \subset \mathcal{L}(X^r)$, $\sum_{k=1}^{\infty} Q_k^0 x = x$ for all $x \in X^r$ and $\dim Q_k^0(X^r) \leq \dim Q_k(X^r \times Y), k \in \mathbb{N}$. Using Theorem 3, we get the corollary. □

By Theorem 3 and the proof of Proposition 2 we obtain

Corollary 5. *The spaces X^r, $r \in \mathbb{N}_0$, are pairwise-nonisomorphic.*

Now we construct a nuclear Fréchet space with a finite-dimensional Schauder decomposition but without a strongly finite-dimensional Schauder decomposition (cf [5]).

Put $\Psi_0 = \bigcup\{\Psi_r : r \in \mathbb{N}_0\}$. Let $\sigma_0 : \mathbb{N} \to \Psi_0$ be an infinity-to-one surjection. Let $n \in \mathbb{N}$. Set $\sigma_0(n) = (p_1, \ldots, p_{r(n)})$ and $p_0 = 0$. Consider the following sequence $(\|\cdot\|^{r(n)}_{n,k})_{k=1}^{\infty}$ of norms on the space $X_n^{r(n)} = \mathbb{K}^{r(n)}$:

$$\|x\|^{r(n)}_{n,k} = \begin{cases} |x|^{r(n)}_{n,j+1} & \text{if } p_j < k \leq p_{j+1}, 0 \leq j \leq r(n)-1, \\ |x|^{r(n)}_{n,r(n)+1} & \text{if } k > p_{r(n)}. \end{cases}$$

Clearly $\|x\|^{r(n)}_{n,k} \leq \|x\|^{r(n)}_{n,k+1}$ for all $x \in X_n^{r(n)}, k \in \mathbb{N}$.

Let $|||x|||_k^0 = \sup_n n^k \|x_n\|_{n,k}^{r(n)}$ for $x = (x_n) \in \prod_{n=1}^{\infty} X_n^{r(n)}, k \in \mathbb{N}$ and

$$X^0 = \left\{ x \in \prod_{n=1}^{\infty} X_n^{r(n)} : |||x|||_k^0 < \infty \text{ for all } k \in \mathbb{N} \right\}.$$

Clearly, $(|||\cdot|||_k^0)$ is a non-decreasing sequence of norms on the linear space X^0.
The proof of Proposition 2 with obvious changes gives the following.

Proposition 6. *The metrizable lcs $X^0 = (X^0, (|||\cdot|||_k^0))$ is a nuclear Fréchet space with a finite-dimensional Schauder decomposition.*

Similarly, the proof of Theorem 3 with slight changes shows the following theorem.

Theorem 7. *For any sequence $(Q_k) \subset \mathcal{L}(X^0)$ such that $\sum_{k=1}^{\infty} Q_k x = x, x \in X^0$, we have $\sup_k \dim Q_k(X^0) = \infty$. In particular, the space X^0 has no strongly finite-dimensional Schauder decomposition.*

Corollary 8. *Let Y be a Fréchet space. For any sequence $(Q_k) \subset \mathcal{L}(X^0 \times Y)$ such that $\sum_{k=1}^{\infty} Q_k z = z, z \in X^0 \times Y$, we have $\sup_k \dim Q_k(X^0 \times Y) = \infty$. In particular, the Fréchet space $X^0 \times Y$ has no strongly finite-dimensional Schauder decomposition.*

Using Proposition 2 and Corollary 4 we obtain the following.

Corollary 9. *The Cartesian product $\prod_{r \in \mathbb{N}_0} X^r$ is a nuclear Fréchet space with a finite-dimensional Schauder decomposition but without a strongly finite-dimensional Schauder decomposition. This space has no continuous norm, so it is not isomorphic to X^0.*

Finally, we construct a nuclear Fréchet space with a Schauder decomposition but without the bounded approximation property, in particular, without a finite-dimensional Schauder decomposition (cf [10]).
Let $\alpha \in \mathbb{K}$ with $0 < |\alpha| < 1$. For $k \in \mathbb{N}$ and $x = (x_{n,p,q}) \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ we put

$$\|x\|_k =$$
$$\max\left(\bigcup_{n,p} \left(\{|x_{n,p,q}|k^{n+p+q} : q \le k\} \cup \{|\alpha^p x_{n,p,q} - x_{n+1,p,q}|k^{n+p+q} : q > k\} \right) \right)$$

and let

$$X = \{x \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}} : \|x\|_k < \infty \text{ for all } k \in \mathbb{N}\}.$$

Note that $(\|\cdot\|_k)$ is a non-decreasing sequence of norms on the linear space X. Indeed, if $x \in X, k \in \mathbb{N}$ and $\|x\|_k = 0$, then $x_{n,p,q} = 0$ if $q \le k$ and $x_{n,p,q} = x_{1,p,q}\alpha^{p(n-1)}$ if $q > k$. Fix $p, q \in \mathbb{N}$ and take $l \in \mathbb{N}$ with $l > \max\{|\alpha|^{-p}, q\}$. Then $|x_{1,p,q}| \max_n (|\alpha^p| l)^n l^{p+q} \le \|x\|_l < \infty$. Hence $x_{1,p,q} = 0$, so $x = 0$.
Since

$\max_{n,p} |\alpha^p x_{n,p,k+1} - x_{n+1,p,k+1}|k^{n+p+k+1} \leq \max_{n,p} |x_{n,p,k+1}|(k+1)^{n+p+k+1}$, then $\|x\|_k \leq \|x\|_{k+1}$ for all $x \in X, k \in \mathbb{N}$.

We have the following

Proposition 10. *The metrizable lcs $X = (X, (\|\cdot\|_k))$ is a nuclear Fréchet space with a Schauder decomposition.*

Proof. First we prove that X is a Fréchet space. Let (x^m) be a Cauchy sequence in X and $x^m = (x^m_{n,p,q})_{n,p,q\in\mathbb{N}}, m \in \mathbb{N}$. Then

$$\forall k \in \mathbb{N}\, \forall \epsilon > 0\, \exists M(\epsilon,k) > 0\, \forall m,l > M(\epsilon,k) : \|x^m - x^l\|_k \leq \epsilon.$$

Fix $n,p,q \in \mathbb{N}$ and $k \geq q$. Then

$$\forall \epsilon > 0\, \forall m,l > M(\epsilon,k) : |x^m_{n,p,q} - x^l_{n,p,q}| \leq \|x^m - x^l\|_k \leq \epsilon,$$

so $(x^m_{n,p,q})_{m=1}^\infty$ is a Cauchy sequence in $(\mathbb{K}, |\cdot|)$. Thus $\lim_m |x^m_{n,p,q} - x^0_{n,p,q}| = 0$ for some $x^0_{n,p,q} \in \mathbb{K}$. Put $x^0 = (x^0_{n,p,q})$. Let $k \in \mathbb{N}$. Then

$$\forall \epsilon > 0\, \forall m,l > M(\epsilon,k)\, \forall n,p \in \mathbb{N}\, \forall q \leq k : |x^m_{n,p,q} - x^l_{n,p,q}|k^{n+p+q} \leq \epsilon.$$

Hence $\forall \epsilon > 0\, \forall m > M(\epsilon,k)\, \forall n,p \in \mathbb{N}\, \forall q \leq k : |x^m_{n,p,q} - x^0_{n,p,q}|k^{n+p+q} \leq \epsilon$.

Similarly, we obtain that $\forall \epsilon > 0\, \forall m > M(\epsilon,k)\, \forall n,p \in \mathbb{N}\, \forall q > k$:

$$|\alpha^p(x^m_{n,p,q} - x^0_{n,p,q}) - (x^m_{n+1,p,q} - x^0_{n+1,p,q})|k^{n+p+q} \leq \epsilon.$$

Hence $\forall \epsilon > 0\, \forall m > M(\epsilon,k) : \|x^m - x^0\|_k \leq \epsilon$. Since $\|x^m\|_k < \infty$, then $\|x^0\|_k < \infty$. Thus $x^0 \in X$, and $\lim_m \|x^m - x^0\|_k = 0$ for all $k \in \mathbb{N}$.

To prove that X is nuclear we show that for all $k \in \mathbb{N}, \epsilon > 0$ there exists a finite subset A^ϵ_k of X such that $B^1_{k+1} \subset B^\epsilon_k + \operatorname{co} A^\epsilon_k$ where $B^1_{k+1} = \{x \in X : \|x\|_{k+1} \leq 1\}$and $B^\epsilon_k = \{x \in X : \|x\|_k \leq \epsilon\}$.

Let $k \in \mathbb{N}, \epsilon > 0$. Choose $m \in \mathbb{N}$ with $[k(k+1)^{-1}]^m < \epsilon$. Put $t = [k(k+1)^{-1}]$.

Let $\{e_{n,p,q} : n,p,q \in \mathbb{N}\}$ be the canonical 'basis' in X. Let $x \in B^1_{k+1}$. For $(p,q) \in \mathbb{N}^2$ we put $x^{(p,q)} = \sum_{n=1}^\infty x_{n,p,q}e_{n,p,q}$; clearly $x^{(p,q)} \in B^1_{k+1}$.

Let $p,q \in \mathbb{N}$ with $p+q < m, q \leq k+1$. Then $\|x^{(p,q)} - \sum_{n=1}^{m-1} x_{n,p,q}e_{n,p,q}\|_k \leq$

$$\max_{n\geq m} |x_{n,p,q}|k^{n+p+q} =$$

$$\max_{n\geq m} |x_{n,p,q}|(k+1)^{n+p+q}t^{n+p+q} \leq \|x^{(p,q)}\|_{k+1}t^m \leq \epsilon,$$

and $\max_n |x_{n,p,q}| \leq \max_n |x_{n,p,q}|(k+1)^{n+p+q} = \|x^{(p,q)}\|_{k+1} \leq 1$.

Hence $x^{(p,q)} \in B^\epsilon_k + \operatorname{co}\{e_{n,p,q} : n < m\}$.

Let $p,q \in \mathbb{N}$ with $p+q < m, q > k+1$. Put

$$y_{n,p,q} = (\alpha^{-np}x_{n,p,q} - \alpha^{-(n+1)p}x_{n+1,p,q}), f_{n,p,q} = \sum_{k=1}^{n} \alpha^{kp}e_{k,p,q}, n \in \mathbb{N}.$$

Since

$$\sum_{n=1}^{m-1} y_{n,p,q} f_{n,p,q} = \sum_{n=1}^{m-1} e_{n,p,q}(x_{n,p,q} - \alpha^{(n-m)p} x_{m,p,q}),$$

then we obtain $\|x^{(p,q)} - \sum_{n=1}^{m-1} y_{n,p,q} f_{n,p,q}\|_k =$

$$\|\sum_{n=1}^{\infty} x_{n,p,q} e_{n,p,q} + x_{m,p,q}\alpha^{-pm}(\sum_{n=1}^{m-1} \alpha^{pn} e_{n,p,q})\|_k =$$
$$\max_{n\geq m} |\alpha^p x_{n,p,q} - x_{n+1,p,q}| k^{n+p+q} =$$
$$\max_{n\geq m} |\alpha^p x_{n,p,q} - x_{n+1,p,q}|(k+1)^{n+p+q} t^{n+p+q} \leq \|x^{(p,q)}\|_{k+1} t^m \leq \epsilon.$$

Moreover we have $\max_n \|y_{n,p,q} f_{n,p,q}\|_{k+1} = \|x^{(p,q)}\|_{k+1} \leq 1$.

Hence $x^{(p,q)} \in B_k^\epsilon + \operatorname{co}\{\beta_{p,q} f_{n,p,q} : n < m\}$ for any $\beta_{p,q} \in \mathbb{K}$ with $\|\beta_{p,q} f_{n,p,q}\|_{k+1} \geq 1$, for $n < m$.

It is easy to check that $\|x - \sum_{p+q<m} x^{(p,q)}\|_k \leq t^m \|x\|_{k+1} \leq \epsilon$. Thus $x \in B_k^\epsilon + \operatorname{co} A_k^\epsilon$ where $A_k^\epsilon = (\{e_{n,p,q} : n, p+q < m, q \leq k+1\} \cup \{\beta_{p,q} f_{n,p,q} : n, p+q < m, q > k+1\})$. Hence $B_{k+1}^1 \subset B_k^\epsilon + \operatorname{co} A_k^\epsilon$.

For $n \in \mathbb{N}$ we put $P_n : X \to X, x \to \sum_{p+q=n} x^{(p,q)}$. Clearly, $P_n, n \in \mathbb{N}$ are continuous linear projections and $P_n P_m = 0$ for $n \neq m$. Since $\|x - \sum_{p+q<m} x^{(p,q)}\|_k \leq [k(k+1)^{-1}]^m \|x\|_{k+1}$ for any $k, m \in \mathbb{N}$, then $\sum_{p+q<m} x^{(p,q)} \to x$ in X, as $m \to \infty$; so $x = \sum_{m=1}^{\infty} P_m x$ for any $x \in X$. Of course $P_n \neq 0, n \in \mathbb{N}$. Thus X has a Schauder decomposition. □

Now we show the following

Theorem 11. *The space X has not the bounded approximation property.*

Proof. Suppose, by contradiction, that there exists a sequence $(A_n) \subset \mathcal{L}(X)$ with $\dim A_n(X) < \infty, n \in \mathbb{N}$, such that $A_n x \to x$ for all $x \in X$. By the Banach-Steinhaus theorem the operators $A_n, n \in \mathbb{N}$, are equicontinuous. Thus there exist $k, l \in \mathbb{N}$ with $k < l$ and a constant $C > 0$ such that

$$\max_n \|A_n x\|_1 \leq C\|x\|_k \text{ and } \max_n \|A_n x\|_{k+1} \leq C\|x\|_l \text{ for all } x \in X.$$

Since $\dim A_n(X) < \infty, n \in \mathbb{N}$, then

$$\forall n \in \mathbb{N}\, \exists C_n > 0\, \forall x \in X : \|A_n x\|_{k+1} \leq C_n \|A_n x\|_1.$$

Let (x^m) be a Cauchy sequence in $(X, \|\cdot\|_l)$ that converges to 0 in $(X, \|\cdot\|_k)$. We show that it converges to 0 in $(X, \|\cdot\|_{k+1})$. Let $\delta > 0$. Then there exist $t, n, s \in \mathbb{N}$ such that $\|x^m - x^t\|_l \leq \delta$ for $m \geq t$, $\|x^t - A_n x^t\|_{k+1} \leq \delta$, and $C_n \|x^m\|_k \leq \delta$ for $m \geq s$. Hence, for $m \geq \max\{t, s\}$ we have

$$\|x^m\|_{k+1} \leq \max\{\|x^m - x^t\|_{k+1}, \|x^t - A_n x^t\|_{k+1}, \|A_n(x^t - x^m)\|_{k+1}, \|A_n x^m\|_{k+1}\} \leq$$
$$\max\{\|x^m - x^t\|_l, \delta, C\|x^t - x^m\|_l, CC_n\|x^m\|_k\} \leq \delta(C+1).$$

Thus $\|x^m\|_{k+1} \to 0$, as $m \to \infty$.

In order to get a contradiction, we take $p \in \mathbb{N}$ with $l|\alpha|^p < 1$ and put

$$x^m = \sum_{n=1}^{m} \alpha^{np} e_{n,p,k+1}, m \in \mathbb{N}.$$

For $t, m \in \mathbb{N}$ with $t < m$, we obtain

$$\|x^m - x^t\|_l = l^{p+k+1} \max_{t+1 \leq n \leq m} (l|\alpha|^p)^n = l^{p+k+1}(l|\alpha|^p)^{t+1}.$$

Thus (x^m) is a Cauchy sequence in $(X, \|\cdot\|_l)$.

Furthermore we have

$\|x^m\|_k = \max_{1 \leq n \leq m} |\alpha^p x^m_{n,p,k+1} - x^m_{n+1,p,k+1}| k^{n+p+k+1} =$

$$|\alpha|^{p(m+1)} k^{m+p+k+1} \leq k^{p+k}(l|\alpha|^p)^{m+1} \to 0, \text{ as } m \to \infty,$$

and

$$\|x^m\|_{k+1} = \max_{1 \leq n \leq m} |\alpha|^{pn}(k+1)^{n+p+k+1} \not\to 0, \text{ as } m \to \infty. \quad \square$$

Corollary 12. *For any Fréchet space Y the space $X \times Y$ has not the bounded approximation property.*

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